1. Let $A(m,n)$ be the statement

$$\text{"}(\exists m \in \mathbb{N})(\exists n \in \mathbb{N})(3m+5n=12)\text{"}$$

We want to determine whether $A(m,n)$ is True or False

$\forall m,n \geq 2, \quad 3m+5n \geq 3(2)+5(2) = 16$

when $m=n=1, \quad 3m+5n = 3(1)+5(1) = 8$
when $m=1, n=2, \quad 3m+5n = 3(1)+5(2) = 13$
when $m=2, n=1, \quad 3m+5n = 3(2)+5(1) = 11$

Thus, $A(m,n)$ is <u>False</u> $\forall m,n \in \mathbb{N}$

2. We want to determine whether the following statement is True or False:

"The sum of any five consecutive integers is divisible by 5 (without remainder)"

Let us represent that statement by S_n

$\therefore S_n = n+(n+1)+(n+2)+(n+3)+(n+4)$ (for an arbitrary $n \in \mathbb{Z}$)

So $S_n = 5n+10 = 5(n+2)$, which is an integer multiple of 5 (i.e. divisible by 5 without remainder)

Thus, S_n is True $\forall n \in \mathbb{Z}$

3. Let $B_n = n^2+n+1 \quad \forall n \in \mathbb{Z}$

we want to determine if B_n is odd or not $\forall n \in \mathbb{Z}$

$B_n = n^2+n+1 = n(n+1)+1$, which is always odd because $n(n+1)$ is always even (because either n is even or $n+1$ is even)

$\therefore$ The original statement is True

4. We want to prove that every odd ~~natural~~ natural number is one of the forms $4n+1$ or $4n+3$ $n \in \mathbb{Z}$

<u>Proof</u>: using divisibility theorem, any integer can be expressed as one of the four forms

$$4n, 4n+1, 4n+2, 4n+3 \qquad n \in \mathbb{Z}$$

$\therefore$ any <u>odd</u> integer (by extension any odd natural number) is one of the 2 forms $4n+1$ or $4n+3$ $n \in \mathbb{Z}$

5. We want to prove that $\forall n \in \mathbb{Z}$, at least one of the integers $n, n+2, n+4$ is divisible by 3

<u>Proof</u>: By Mathematical Induction

Case $n=1$: at least one of $1, 3, 5$ is divisible by 3 ✓

assume true for $n=k$: at least one of $k, k+2, k+4$ is divisible by 3

Now, for $n=k+1$: $n=k+1$, $n+2=k+3$, $n+4=k+5$

if k is divisible by 3, so is $k+3$

if $k+2$ is divisible by 3, so is $k+5$ (i.e. $(k+2)+3$)

if $k+4$ is divisible by 3, so is $k+1$ (i.e. $(k+4)-3$)

$\therefore$ at least one of $k+1, k+3, k+5$ is divisible by 3 (using induction hypothesis)

Thus the statement is true for $n=1$, and it is true for $n=k+1$ (given that it is true for $n=k$). So it is true $\forall n \in \mathbb{Z}$ by the principle of mathematical induction. ✓

6. Using the results from question # 5,
i.e. $\forall n$ at least one of the integers $n, n+2, n+4$ is divisible by 3.

From this result, $\forall n > 3$, $n, n+2, n+4$ is not a prime triple because at least one of the 3 numbers is divisible by 3

$\forall n < 3$, the same argument holds as $\forall n > 3$

For $n = 3$, $3, 5, 7$ is a prime triple (and the <u>only</u> one)

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7. To prove $\sum_{k=1}^{n} 2^k = 2^{n+1} - 2 \quad \forall n \in \mathbb{N}$

<u>Proof</u>: by the principle of mathematical induction

Case $n=1$, $2^1 = 2^{1+1} - 2 \Rightarrow 2 = 2$ ✓

assume true for $n = p$ i.e. $\sum_{k=1}^{p} 2^k = 2^{p+1} - 2$

Now for $n = p+1$,

$$\sum_{k=1}^{p+1} 2^k = \sum_{k=1}^{p} 2^k + 2^{p+1}$$
$$= \left(2^{p+1} - 2\right) + 2^{p+1} \quad \text{(by induction hypothesis)}$$
$$= 2\left(2^{p+1}\right) - 2$$
$$= 2^{p+2} - 2 \quad ✓$$

∴ Thus the identity is true for $n=1$, and it is true for $n = p+1$ (given that it is true for $n = p$). So it is true $\forall n \in \mathbb{N}$ by the principle of mathematical induction. ✓

8. $\{a_n\}_{n=1}^{\infty} \to L$ as $n \to \infty$

$\Leftrightarrow (\forall \epsilon > 0)(\exists n_0)(\forall n \geq n_0)\left[|a_n - L| < \epsilon\right]$

Now $|a_n - L| < \epsilon \Rightarrow |Ma_n - ML| < M\epsilon \quad (M > 0)$

$\Rightarrow |Ma_n - ML| < \epsilon'$ (where $\epsilon' = M\epsilon$)

$\therefore (\forall \epsilon' > 0)(\exists n_0)(\forall n \geq n_0)\left[|Ma_n - ML| < \epsilon'\right]$

$\Leftrightarrow \boxed{\{Ma_n\}_{n=1}^{\infty} \to ML \text{ as } n \to \infty}$

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9. Example: $A_n = (0, \frac{1}{n})$ $n = 1, 2, \ldots$

$A_1 = (0, 1)$, $A_2 = (0, \frac{1}{2})$, $A_3 = (0, \frac{1}{3})$, ...

So, $A_2 \subset A_1$, $A_3 \subset A_2$, ... $A_{n+1} \subset A_n$ ✓

as $n \to \infty$ $\frac{1}{n} \to 0$, and $(0, \frac{1}{n}) \to \emptyset$

So $\bigcap_{n=1}^{\infty} A_n = \emptyset$ ✓

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10 Example: $A_n = [0, \frac{1}{n}]$ $n = 1, 2, \ldots$

$A_1 = [0, 1]$, $A_2 = [0, \frac{1}{2}]$, $A_3 = [0, \frac{1}{3}]$, ...

So $A_2 \subset A_1$, $A_3 \subset A_2$, ... $A_{n+1} \subset A_n$ ✓

as $n \to \infty$, $\frac{1}{n} \to 0$, and $[0, \frac{1}{n}] \to \{0\}$

So. $\bigcap_{n=1}^{\infty} A_n = \{0\}$ ✓